Document models

Ayush Tewari

November 24th, 2025

Adapted from Carl Edward Rasmussen

Key concepts

- a simple document model

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- a simple document model
- a mixture model for document

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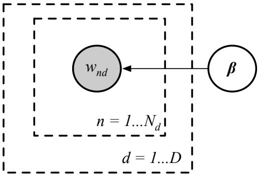
- a simple document model
- a mixture model for document
- fitting the mixture model with EM

A really simple document model

Consider a collection of D documents from a vocabulary of M words.

- N_d: number of words in document d.



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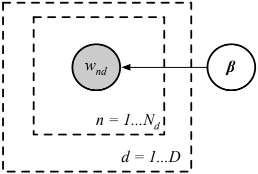

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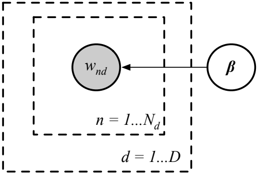

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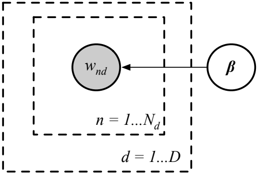
- $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_M]^\top$: parameters of a categorical / multinomial distribution[1] over the M vocabulary words.



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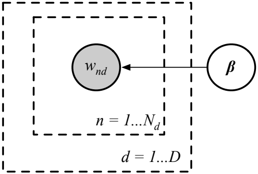

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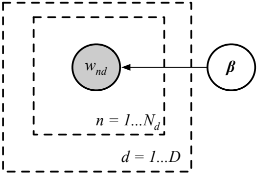

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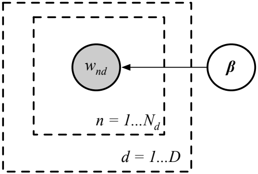

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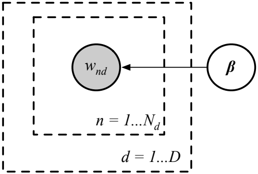

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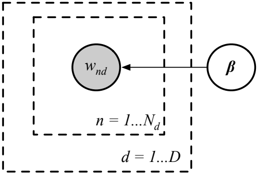

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$$\boxed{\hat{\beta}_m = \frac{c_m}{N} = \frac{c_m}{\sum_{\ell=1}^{M} c_\ell}}$$

- $N = \sum_{d=1}^{D} N_d$: total number of words in the collection.
- $c_m = \sum_{d=1}^{D} \sum_{n=1}^{N_d} \mathbb{I}(w_{nd} = m)$: total count of vocabulary word m.

Maximum Likelihood and Lagrange multipliers

In maximum likelihood learning, we want to maximize the (log) likelihood

- Likelihood: $p(\mathrm{w}|\boldsymbol{\beta}) = \prod_{d=1}^{D} \prod_{n=1}^{N_d} \beta_{w_{nd}} = \prod_{m=1}^{M} \beta_m^{c_m}$

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which we combine to $\beta_m = c_m/N$, where N is the total number of words.

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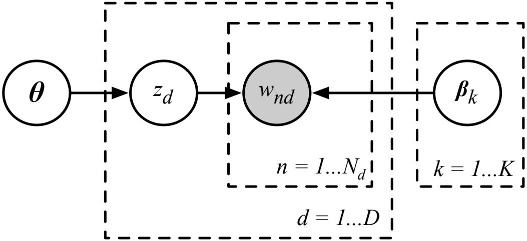
- **This generative model does not specialise.**
- We would like a model where different documents might be about different *topics*.

A mixture of categoricals model



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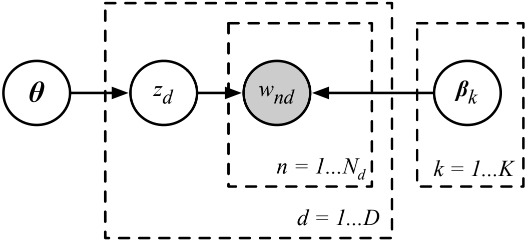

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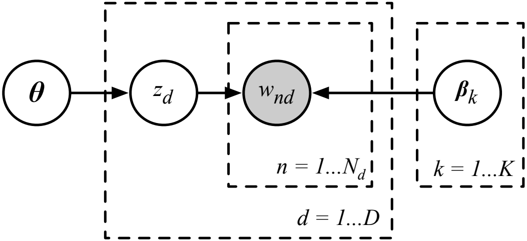

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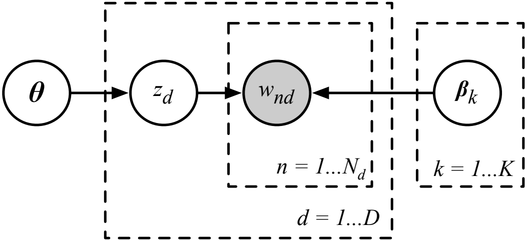

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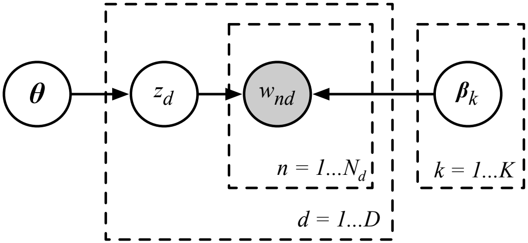

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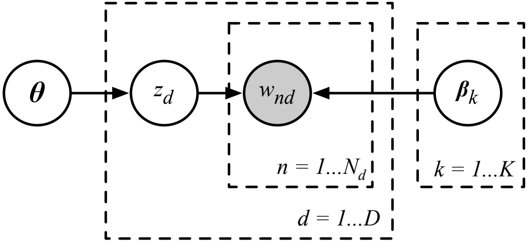

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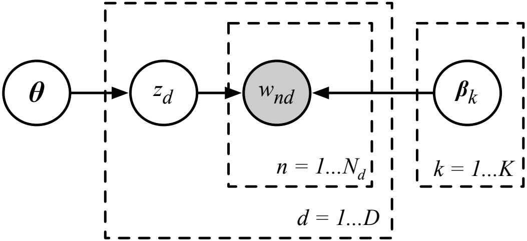

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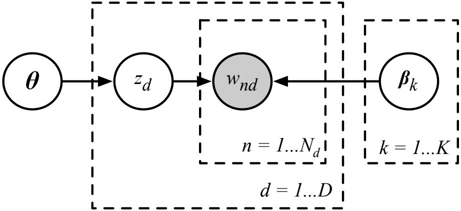
We have introduced a new set of *hidden* variables z_d.

- How do we fit those variables? What do we do with them?
- Are these variables interesting? Or are we only interested in $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$?

A mixture of categoricals model: the likelihood



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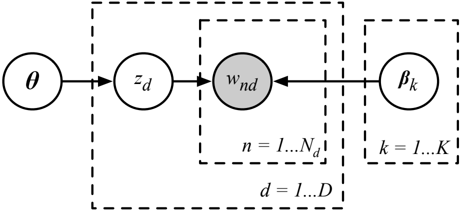

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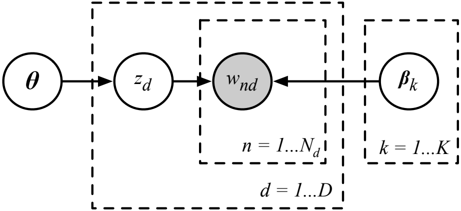

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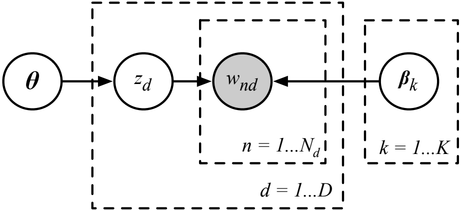

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EM and Mixtures of Categoricals

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E-step: for each d, set q to the posterior (where $c_{md} = \sum_{n=1}^{N_d} \mathbb{I}(w_{nd} = m)$):

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M-step: Maximize

$$\begin{aligned}\sum_{d=1}^{D}\sum_{k=1}^{K} q(z_d = k) \log p(\mathrm{w}, z_d) &= \sum_{k,d} r_{kd} \log\left[p(z_d = k|\theta) \prod_{n=1}^{N_d} p(w_{nd}|\beta_{k,w_{nd}})\right] \\ &= \sum_{k,d} r_{kd}\left(\log \prod_{m=1}^{M} \beta_{km}^{c_{md}} + \log\theta_k\right) \\ &= \sum_{k,d} r_{kd}\Big(\sum_{m=1}^{M} c_{md} \log \beta_{km} + \log\theta_k\Big) \;\overset{\text{def}}{=}\; F(R, \theta, \beta)\end{aligned}$$

EM: M step for mixture model

$$F(R, \theta, \beta) = \sum_{k,d} r_{kd} (\sum_{m=1}^{M} c_{md} \log \beta_{km} + \log \theta_k)$$

Need Lagrange multipliers to constrain the maximization of F and ensure proper distributions.

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Update for θ (Topic Proportions):

$$\hat{\theta}_k \leftarrow \mathrm{argmax}_{\theta_k}\ \ F(R,\theta,\beta) + \lambda\big(1 - \sum_{k'=1}^{K}\theta_{k'}\big)$$

$$= \frac{\sum_{d=1}^{D} r_{kd}}{\sum_{k'=1}^{K}\sum_{d=1}^{D} r_{k'd}} = \boxed{\frac{\sum_{d=1}^{D} r_{kd}}{D}}$$

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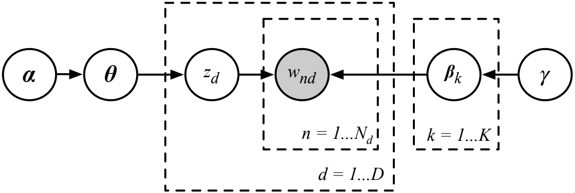
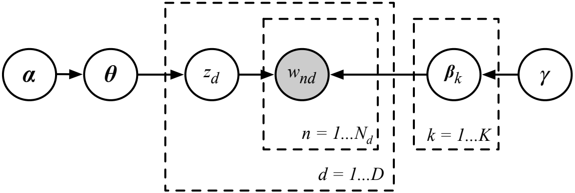
Update for β **(Word Probabilities):**

$$\hat{\beta}_{km} \leftarrow \mathrm{argmax}_{\beta_{km}} \; F(R, \theta, \beta) + \sum_{k'=1}^{K} \lambda_{k'}(1 - \sum_{m'=1}^{M} \beta_{k'm'})$$

A Bayesian mixture of categoricals model



A Bayesian mixture of categoricals model



$$\begin{aligned}
\boldsymbol{\theta} &\sim \mathrm{Dir}(\alpha) \\
\boldsymbol{\beta}_k &\sim \mathrm{Dir}(\gamma) \\
z_d | \boldsymbol{\theta} &\sim \mathrm{Cat}(`) \\
w_{nd} | z_d, \boldsymbol{\beta} &\sim \mathrm{Cat}(\mathrm{fi}_{z_d})
\end{aligned}$$

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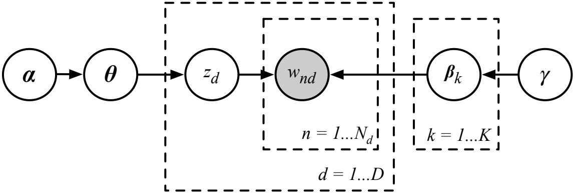

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- With the EM algorithm we have essentially estimated θ and β by maximum likelihood.

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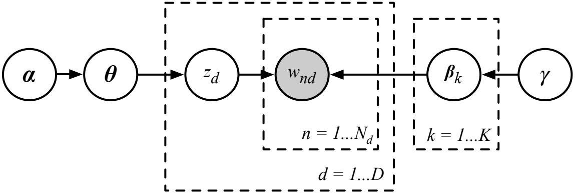

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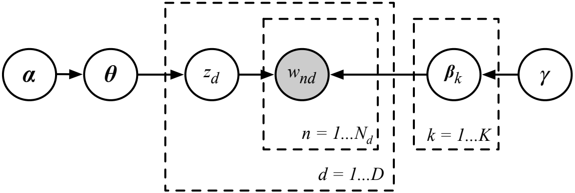
- With the EM algorithm we have essentially estimated $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ by maximum likelihood.
- An alternative, Bayesian treatment infers these parameters starting from priors:
  - $\boldsymbol{\theta} \sim \mathrm{Dir}(\alpha)$ is a symmetric Dirichlet over category probabilities.
  - $\boldsymbol{\beta}_k \sim \mathrm{Dir}(\gamma)$ are symmetric Dirichlets over vocabulary probabilities.

A Bayesian mixture of categoricals model



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What is different?

- We no longer want to compute a point estimate of θ or β.
- We are now interested in computing the *posterior* distributions.